Math 279 Lecture 15 Notes

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1 Bounds for Germs

1.1 Condition for coherence of germs

Ultimately, we wish to find a distribution $u \in \mathcal{D}'$ that is well-approximated by a germ. Recall that a **germ** is $F : \mathbb{R}^d \to \mathcal{D}'$ that is measurable.

Proposition 1.1. *Let F be a germ, and assume that there exists a constant c, a compact set K, exponents γ and r, and a distribution u such that*

$$|(u - F_x)(\phi_x^\delta)| \leq c\delta^\gamma$$

for all $x \in K$, $\delta \in (0,1]$, and $\phi \in \mathcal{D}$ such that $\operatorname{supp}\phi \subseteq B_1(0) = \{x : |x| \leq 1\}$ and $\|\phi\|_{C^r} \leq 1$. (Here, $F_x := F(x)$.) Then

$$|(F_x - F_y)(\phi_y^\delta)| \leq 2c\delta^{-\tau}(|x-y| + \delta)^{\gamma+\tau},$$

provided that $\delta \in (0, 1/2]$, ϕ is as before, and $|x-y| \leq 1/2$. Here, we may choose $\tau = d+r$.

Remark 1.1. This basically says that F is (τ,γ)-coherent. Observe that

$$\delta^{-\tau}(|x-y|+\delta)^{\gamma+\tau} \lesssim \begin{cases} \delta^{-\tau}|x-y|^{\gamma+\tau} = \delta^\gamma(\frac{|x-y|}{\delta})^{\gamma+\tau} & |x-y| > \delta \\ \delta^\gamma & |x-y| < \delta, \end{cases}$$

so this second case is an improvement. Also observe that if F is (τ,γ)-coherent and $\tau \leq \tau'$, then it is also (τ',γ)-coherent.

Proof. Observe that we can write

$$\begin{aligned} |(F_x - F_y)(\phi_y^\delta)| &\leq |(F_x - u)(\phi_y^\delta)| + |(u - F_y)(\phi_y^\delta)| \\ &\leq |(F_x - u)(\phi_y^\delta)| + c \underbrace{\delta^\gamma}_{\leq \delta^\tau(|x-y|+\delta)^{\gamma+\tau}}. \end{aligned}$$

It remains to bound $|(u - F_x)(\phi_y^\delta)|$. Note that $x \neq y$ in general. Observe that

$$\begin{aligned}
\phi_y^\delta(z) &= \frac{1}{\delta^d}\phi\left(\frac{z-y}{\delta}\right) \\
&= \frac{1}{\delta^d}\phi\left(\frac{(z-x)-(y-x)}{\delta}\right) \\
&= \frac{1}{\delta^d}\phi\left(\frac{z-x-\frac{y-x}{|y-x|+\delta}(|y-x|+\delta)}{(|y-x|+\delta)\frac{\delta}{|y-x|+\delta}}\right)
\end{aligned}$$

Denote $\varepsilon := |y-x|+\delta$, $\varepsilon' := \frac{\delta}{|y-x|+\delta}$, and $a := \frac{y-x}{|y-x|+\delta}$.

$$\begin{aligned}
&= \frac{1}{\varepsilon^d}\frac{1}{(\varepsilon')^d}\phi\left(\frac{\frac{z-x}{\varepsilon}-a}{\varepsilon'}\right) \\
&= \frac{1}{\varepsilon^d}\phi_a^{\varepsilon'}\left(\frac{z-x}{\varepsilon}\right)
\end{aligned}$$

Denote $\psi := \phi_a^{\varepsilon'}$.

$$= \psi_\varepsilon^x(z).$$

Now observe that by definition, our new test function

$$\psi(z) = (\varepsilon')^{-d}\phi\left(\frac{z-a}{\varepsilon'}\right), \qquad a = \frac{y-x}{|y-x|+\delta},$$

so let's examine the support of ψ: $\operatorname{supp}\psi \subseteq B_{\varepsilon'+a}(0)$, if $\operatorname{supp}\phi \subseteq B_1(0)$. Note that $\varepsilon' + |a| = \frac{|y-x|}{|y-x|+\delta} + \frac{\delta}{|y-x|+\delta} = 1$.

We can also rephrase the condition of $\|\phi\|_{C^r} \leq 1$ as just giving a factor of $\|\phi\|_{C^r}$ in the inequality in the hypothesis of the theorem.

We can now argue that

$$\begin{aligned}
|(u-F_x)(\phi_y^\delta)| &= |(u-F_x)(\psi_x^\varepsilon)| \\
&\leq c\varepsilon^\gamma \|\psi\|_{C^r}.
\end{aligned}$$

On the other hand, $\|\psi\|_{C^r} \leq (\varepsilon')^{-(d+r)}$. Hence,

$$\begin{aligned}
|(u-F_x)(\phi_y^\delta)| &\leq c(|y-x|+\delta)^\gamma \left(\frac{\delta}{|y-x|+\delta}\right)^{-d-r} \\
&= c\delta^{-\tau}(|y-x|+\delta)^{\gamma+\tau},
\end{aligned}$$

where $\tau = d + r$, as desired. □

1.2 Uniform bounds on germs

We now address the following question: Assume that

$$\delta^{-\gamma} \sup_{x\in K} \sup_{\|\phi\|_{C^r}\le 1} (u - F_x)(\phi_x^\delta) \le c$$

for $\delta \in (0,1]$.

Proposition 1.2. *Suppose $F = (F_x : x \in \mathbb{R}^d)$ is a $(-\tau,\gamma)$-coherent germ with respect to ϕ.*[1] *Then there exists $\eta = \eta_K$ such that $|F_x(\phi_x^\delta)| \lesssim \delta^{-\eta}$ uniformly in a compact set K and uniformly for $\delta \in [0,1]$.*

The important part is that we can choose η independent of x.

Proof. Fix $a \in K$, and observe that

$$|F_a(\phi_x^\delta)| \le c_0 \|\phi_x^\delta\|_{C^r} \le c_1 \delta^{-d-r} \|\phi\|_{C^r}$$

for ϕ such that $\operatorname{supp} \phi_x^\delta$ is in some compact set. This is the case if $\delta \in (0,1]$ and $x \in K$. We now use the coherence to assert that for $x \in K$,

$$\begin{aligned} |F_x(\phi_x^\delta)| &\le |(F_a - F_x)(\phi_x^\delta)| + |F_a(\phi_x^\delta)| \\ &\le c\delta^{-\tau}(\underbrace{|a-x|}_{\operatorname{diam} K} + \delta)^{\gamma+\tau} + c_2 \delta^{-d-r}. \end{aligned}$$

We are done if we choose $\eta = \max\{\tau, d+r\}$. $\square$

1.3 Preparation for proving the reconstruction theorem

We now focus on the proof of the reconstruction theorem of Hairer.[2] As a preparation, we start with a test function ϕ with $\int \phi \neq 0$ and switch to a new test function $\widehat{\phi}$ so that

$$\int \widehat{\phi} = \int \phi, \quad \text{but} \int \widehat{\phi}(x) x^r \, dx = 0 \quad \text{for} \quad 0 < |r| \le \ell - 1.$$

In fact, what we have in mind is

$$\widehat{\phi} = \sum_{i=0}^{\ell-1} c_i \phi^{\lambda_i}, \qquad \phi^\lambda(x) := \lambda^{-d} \phi\left(\frac{x}{\lambda}\right).$$

[1] Later, we will see that coherence with respect to 1 ϕ implies coherence with respect to other functions.

[2] Hairer's original proof used wavelets, which we will not use.

In other words, given distinct positive $\lambda_0, \ldots, \lambda_{\ell-1}$, we can find $c_0, \ldots_{\ell+1}$ such that for $\widehat{\phi}$ defined this way, the integral conditions hold. Indeed,

$$\int \widehat{\phi} = \sum_{i=0}^{\ell-1} c_i \int \phi$$

$$\begin{aligned}\int \widehat{\phi} x^r \, dx &= \sum_{i=1} c_i \int x^r \phi^{\lambda_i}(x) \, dx \\ &= \sum_{i=0}^{\ell-1} c_i \lambda_i^{|r|} \int x^r \phi(x) \, dx.\end{aligned}$$

So we need

$$\begin{cases} \sum_{i=0}^{\ell-1} c_i = 1, \\ \sum_{i=0}^{\ell-1} c_i \lambda_i^r \text{ for } r = 1, \ldots, \ell-1, \end{cases} \qquad \text{or} \qquad \underbrace{\begin{bmatrix} 1 & \cdots & 1 \\ \lambda_0 & \cdots & \lambda_{\ell-1} \\ \vdots & & \vdots \\ \lambda_0^{\ell-1} & \cdots & \lambda_{\ell-1}^{\ell-1} \end{bmatrix}}_{A} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{\ell-1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

In fact, there is an explicit formula for A^{-1}, and the answer is

$$c_i = \prod_{j \neq i} \frac{\lambda_j}{\lambda_j - \lambda_i}.$$

Note that we may choose the λ_is small enough so that $\operatorname{supp} \widehat{\phi} \subseteq B(0, 1/2)$.

Out of this $\widehat{\phi}$, we now build another test function of the form

$$\widetilde{\phi} = \widehat{\phi}^2 - \widehat{\phi}^{1/2}.$$

We will use this to prove the reconstruction theorem next time.